Sometimes it's desirable to say an event U has **no** probability value assigned. The **inner** and **outer measures** associated with a single measure μ give values for *all* events U, even where μ is undefined:

$\mu_*(U)$ = **maximum** μ measure of all U's **subsets**

$\mu^*(U)$ = **minimum** μ measure of all U's **supersets**

Treat the *interval* $[\mu_*(U), \mu^*(U)]$ as a surrogate for some unavailable number "probability of U."

EXAMPLE

A ball is drawn from an urn containing 30 red, 70 blue-or-yellow balls.

$$W = \{ \emptyset, \{r\}, \{b\}, \{y\}, \{r,b\}, \{r,y\}, \{b,y\}, \{r,b,y\} \}.$$

It's meaningful to define a probability measure μ only on a *sub*algebra of W:

$$\{ \emptyset, \{r\}, \{b,y\}, \{r,b,y\} \}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$0 \quad 0.3 \quad 0.7 \quad 1$$

$$\mu_*(\{r,y\}) = \max\{\mu(\emptyset), \mu(\{r\})\} = \max\{0, 0.3\} = 0.3$$

$$\mu^*(\{r,y\}) = \min\{\mu(\{r,b,y\})\} = \min\{1\} = 1$$

Sometimes it's desirable to say an event U has **many** probability values assigned. The **lower** and **upper probabilities** associated with a given set of measures $\wp = \{\mu_1, \ldots \mu_n\}$ give single values for *all* events U:

$\wp_*(U)$ = **minimum** of the values assigned to U by measures in $\wp$

$\wp^*(U)$ = **maximum** of the values assigned to U by measures in $\wp$

Treat the *interval* $[\wp_*(U), \wp^*(U)]$ as a surrogate for some unavailable number "probability of U."

EXAMPLE

A ball is drawn from an urn containing 30 red, 70 blue-or-yellow balls.
Let's define a family of $\wp = \{\mu_0, \ldots \mu_{70}\}$ probability measures on these events.

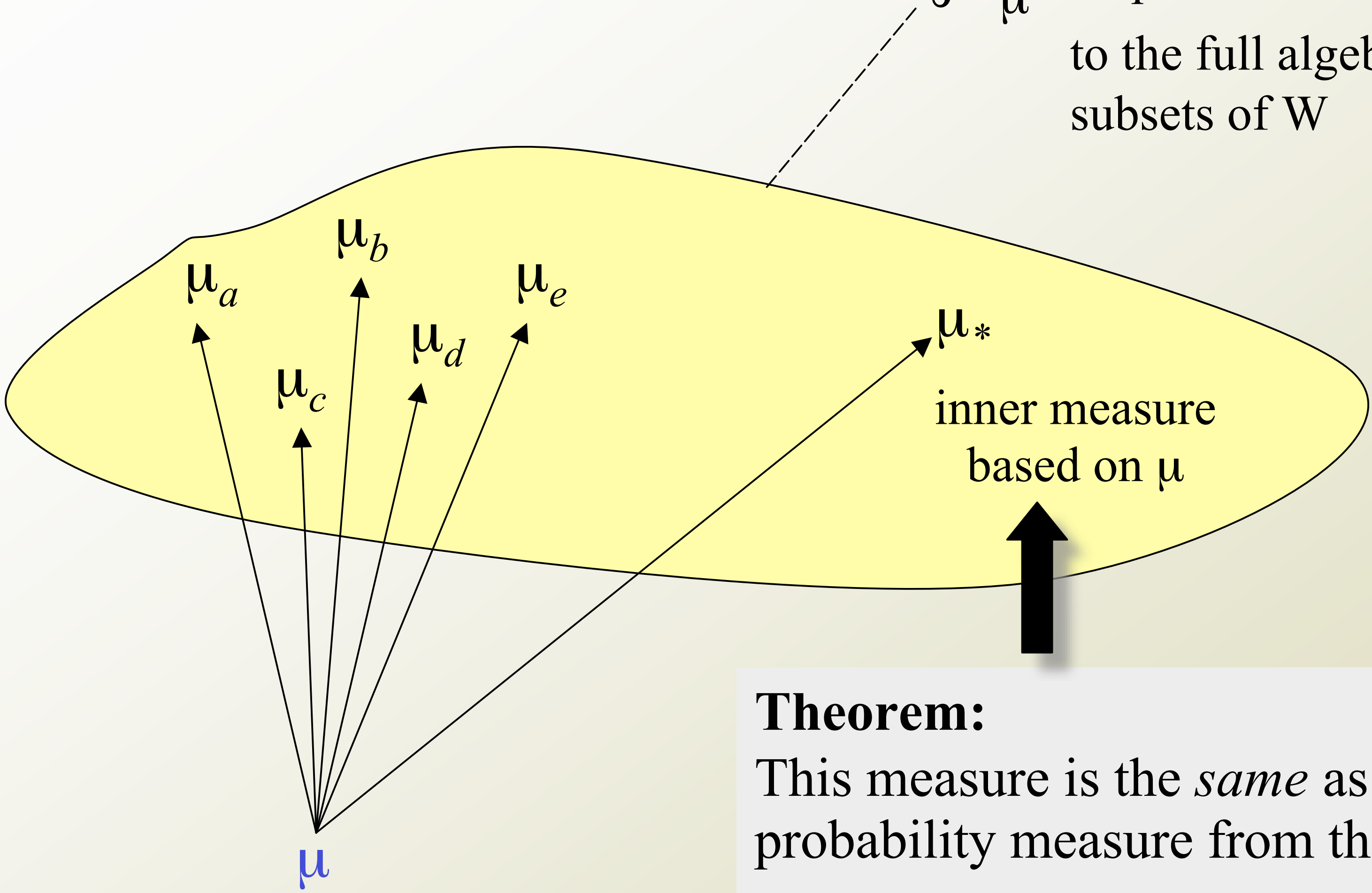
$$W = \{\ \emptyset,\ \{r\},\ \{b\},\ \{y\},\ \{r,b\},\ \{r,y\},\ \{b,y\},\ \{r,b,y\}\ \}.$$

	$\emptyset$	$\{r\}$	$\{b\}$	$\{y\}$	$\{r,b\}$	$\{r,y\}$	$\{b,y\}$	$\{r,b,y\}$
μ_i :	0	0.3	$i/100$	$0.7 - i/100$	$0.3 + i/100$	$1 - i/100$	0.7	1

$$\wp_*(\{r,y\}) = \min\{\mu_i(\{r,y\})\ |i=0\ldots70\} = \min\{1 - i/100\ |i=0\ldots70\} = 0.3$$

$$\wp^*(\{r,y\}) = \max\{\mu_i(\{r,y\})|i=0\ldots70\} = \max\{1 - i/100\ |i=0\ldots70\} = 1$$

$\wp_\mu$: all possible **extensions** of μ to the full algebra of *all* subsets of W



defined on a *sub*algebra of W

Theorem:
This measure is the *same* as the lower probability measure from the set $\wp_\mu$.

So: Any inner measure results from a lower probability. [But not conversely.]

(To get an inner measure μ_*, form the set of all of μ's extensions and take the lower prob.)